

On bipartite cages of excess 4

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Abstract

The Moore bound $M(k, g)$ is a lower bound on the order of k -regular graphs of girth g (denoted (k, g) -graphs). The excess e of a (k, g) -graph of order n is the difference $n - M(k, g)$. In this paper we consider the existence of (k, g) -bipartite graphs of excess 4 via studying spectral properties of their adjacency matrices. We prove that the (k, g) -bipartite graphs of excess 4 satisfy the equation $kJ = (A + kI)(H_{d-1}(A) + E)$, where A denotes the adjacency matrix of the graph in question, J the $n \times n$ all-ones matrix, E the adjacency matrix of a union of vertex-disjoint cycles, and $H_{d-1}(x)$ is the Dickson polynomial of the second kind with parameter $k - 1$ and of degree $d - 1$. We observe that the eigenvalues other than $\pm k$ of these graphs are roots of the polynomials $H_{d-1}(x) + \lambda$, where λ is an eigenvalue of E . Based on the irreducibility of $H_{d-1}(x) \pm 2$ we give necessary conditions for the existence of these graphs. If E is the adjacency matrix of a cycle of order n we call the corresponding graphs *graphs with cyclic excess*; if E is the adjacency matrix of a disjoint union of two cycles we call the corresponding graphs *graphs with bicyclic excess*. In this paper we prove the non-existence of (k, g) -graphs with cyclic excess 4 if $k \geq 6$ and $k \equiv 1 \pmod{3}$, $g = 8, 12, 16$ or $k \equiv 2 \pmod{3}$, $g = 8$, and the non-existence of (k, g) -graphs with bicyclic excess 4 if $k \geq 7$ is odd number and $g = 2d$ such that $d \geq 4$ is even.

Keywords: cage problem, bipartite graphs, cyclic excess, bicyclic excess

1 Introduction

A k -regular graph of girth g is called a (k, g) -graph. A (k, g) -cage is a (k, g) -graph with the fewest possible number of vertices, among all (k, g) -graphs. The order of a (k, g) -cage is denoted by $n(k, g)$. The *Cage Problem* calls for finding cages, and this problem was considered for the first time by Tutte [16]. It is known that a (k, g) -graph

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exists for any combination of $k \geq 2$ and $g \geq 3$, [7, 14]. However, the orders $n(k, g)$ of (k, g) -cages have only been determined for very limited sets of parameters [9]. A natural lower bound on the order of a (k, g) -graph is called the *Moore bound*, and the form of the bound depends on the parity of g , i.e.,

$$n(k, g) \geq M(k, g) = \begin{cases} 1 + k + k(k-1) + \dots + k(k-1)^{(g-3)/2}, & g \text{ odd}, \\ 2(1 + (k-1) + \dots + (k-1)^{(g-2)/2}), & g \text{ even}. \end{cases} \quad (1)$$

The graphs whose orders are equal to the Moore bound are called *Moore graphs*. They are known to exist if $k = 2$ and $g \geq 3$, $g = 3$ and $k \geq 2$, $g = 4$ and $k \geq 2$, $g = 5$ and $k = 2, 3, 7$, or $g = 6, 8, 12$ and a generalized n -gon of order $k - 1$ exists [1, 4, 9]. The existence of a $(57, 5)$ -Moore graph is an open question. The *excess* e of a (k, g) -graph is the difference between its order n and the Moore bound $M(k, g)$, i.e., $e = n - M(k, g)$. Regarding graphs of even girth we will use the following three results:

Theorem 1.1 ([3]) *Let G be a (k, g) -cage of girth $g = 2d \geq 6$ and excess e . If $e \leq k - 2$, then e is even and G is bipartite of diameter $d + 1$.*

For the next theorem, let $D(k, 2)$ denote the incidence graph of a symmetric $(v, k, 2)$ -design.

Theorem 1.2 ([3]) *Let G be a (k, g) -cage of girth $g = 2d \geq 6$ and excess 2. Then $g = 6$, G is a double-cover of $D(k, 2)$, and k is not congruent to 5 or 7 (mod 8).*

Theorem 1.3 ([11]) *Let $k \geq 6$, $g = 2d > 6$. No (k, g) -graphs of excess 4 exist for parameters k, g satisfying at least one of the following conditions:*

- 1) $g = 2p$, with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2 \pmod{p}$;
- 2) $g = 4 \cdot 3^s$ such that $s \geq 4$, and k is divisible by 9 but not by 3^{s-1} ;
- 3) $g = 2p^2$ with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2 \pmod{p}$ and even;
- 4) $g = 4p$, with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2, 3, p - 2 \pmod{p}$;
- 5) $g \equiv 0 \pmod{16}$, and $k \equiv 3 \pmod{g}$.

Motivated by the result in Theorem 1.3, which was obtained through counting cycles in a hypothetical graph with given parameters and excess 4, in this paper we address the question of the existence of (k, g) -graphs of excess 4 using spectral properties of their adjacency matrices. The question of the existence of (k, g) -graphs of excess 4 is wide open, and prior to the publication of [11], no such results were known. The results contained in our paper further extend our understanding of the structure of the potential graphs of excess 4. Throughout, we assume that $k \geq 6$, $g = 2d \geq 6$ and G is a (k, g) -graph of excess 4 and order n . Due to Biggs's result stated in Theorem 1.1, the restriction of the parameters k, g given above allows us

to conclude that G is a bipartite graph with diameter $d + 1$. For each integer i in the range $0 \leq i \leq d + 1$, we define the $n \times n$ matrix $A_i = A_i(G)$ as follows. The rows and columns of A_i correspond to the vertices of G , and the entry in position (u, v) is 1 if the distance $d(u, v)$ between the vertices u and v is i , and zero otherwise. Clearly, $A_0 = I$, $A_1 = A$, the usual adjacency matrix of G . The last non-zero matrix is the matrix A_{d+1} which we shall denote by E and refer to it as the *excess matrix* i.e., E is the adjacency matrix of the graph with the same vertex set V as G such that two vertices of V are adjacent if and only if they have distance $d + 1$. We will call this graph the *excess graph* of G and we will denote it by $G(E)$. If J is the all-ones matrix, the sum of the i -distance matrices A_i , $0 \leq i \leq d$, and the matrix E yields $\sum_{i=0}^d A_i + E = J$. To apply the last identity we will use Lemma 4 from [11]. Employing the methodology used by Bannai et al. in [1], [2], later by Biggs et al. in [3], Delorme et al. in [5] and Garbe in [10], we will show that the eigenvalues of G other than $\pm k$ are the roots of the polynomials $H_{d-1}(x) + \lambda$. Here, $H_{d-1}(x)$ is the Dickson polynomial of the second kind with parameter $k - 1$ and degree $d - 1$, and λ is an eigenvalue of the excess matrix E . Furthermore, for odd $k \geq 7$ and $d \geq 4$, we prove that the polynomial $H_{d-1}(x) \pm 2$ is irreducible over $\mathbb{Q}[x]$, which leads to necessary conditions for existence of (k, g) -graphs of excess 4, Theorem 2.7.

We say that a graph G has a *cyclic excess* if the excess graph $G(E)$ is a cycle of length n , and a graph G has a *bicyclic excess* if $G(E)$ is a disjoint union of two cycles. In [6] Delorme et al. considered graphs with cyclic defect and excess 2, proving non-existence of infinitely many such graphs. The paper describes the cycle structure of the excess graphs of the known non-trivial graphs of excess 2:

- 1) the excess graph of the only $(3, 5)$ -graph of excess 2 is a disjoint union of a 9-cycle and a 3-cycle or a disjoint union of an 8-cycle and 4-cycle;
- 2) the excess graph of the unique $(4, 5)$ -graph of excess 2 (the Robertson graph) is a disjoint union of a 3-cycle, a 12-cycle and a 4-cycle;
- 3) the excess graph of the unique $(3, 7)$ -graph of excess 2 (the McGee graph) is a disjoint union of six 4-cycles.

We note that no (k, g) -graph of cyclic excess 2 are known, while examples of graphs with bicyclic excess 2 can be found among the $(3, 5)$ -graphs of excess 2. Proving that the excess graphs of bipartite graphs of excess 4 form a disjoint union of cycles, while also inspired by the results in [6], in Section 3 we consider the existence of bipartite graphs of excess 4 with cyclic and bicyclic excess 4. Based on the irreducibility of $H_{d-1}(x) \pm 2$ and $H_{d-1}(x) - 1$ over $\mathbb{Q}[x]$, we prove the non-existence of infinitely many such graphs of girths at least 8.

2 Necessary conditions for the existence of graphs of even girth and excess 4

Let $k \geq 6$, $g = 2d \geq 6$, and let G be a (k, g) -graph of excess 4. Then G is bipartite of diameter $d + 1$. Let $N_G(u, i)$ denote the set of vertices of G whose distance from u in G is equal to i , $1 \leq i \leq d + 1$. The subgraph of G induced by the set of vertices of G whose distance from u is at most $\frac{g-2}{2}$ and whose distance from v is by one larger than their distance from u induces a tree of depth $\frac{g-2}{2}$ rooted at u (we will call it \mathcal{T}_u). Also, the subgraph of G induced by the set of vertices of G whose distance from v is at most $\frac{g-2}{2}$ and whose distance from u is by one larger than their distance from v induces a tree of depth $\frac{g-2}{2}$ rooted at v (we will call it \mathcal{T}_v). Since G is of girth g , the trees \mathcal{T}_u and \mathcal{T}_v are disjoint and contain no cycles. Since each vertex of G is of degree k , the order of $\mathcal{T}_u \cup \mathcal{T}_v$ is equal to $2(1 + (k-1) + (k-1)^2 + \dots + (k-1)^{\frac{g-2}{2}})$. We will call the union of the trees $\mathcal{T}_u, \mathcal{T}_v$ with the edge f *Moore tree of G rooted at f* ; it is the subtree of G that is the basis of the Moore bound for even g . The graph G must contain 4 additional vertices w_1, w_2, w_3, w_4 which do not belong to either \mathcal{T}_u or \mathcal{T}_v , and whose distance from both u and v is greater than $\frac{g-2}{2}$. We will call these vertices *the excess vertices with respect to f* and denote this set $X_f = \{w_1, w_2, w_3, w_4\}$; we call the edges not contained in the Moore tree of G *horizontal edges*.

The following lemma restricts the possible ways in which the four excess vertices are attached to the Moore tree.

Lemma 2.1 ([11]) *Let $k \geq 6$, $g = 2d \geq 6$. Let G be a (k, g) -graph of excess 4, u, v be two adjacent vertices in G , and $X_f = \{w_1, w_2, w_3, w_4\}$ be the four excess vertices with respect to the edge $f = \{u, v\}$. The induced subgraph $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$ (two disjoint copies of K_2) or \mathcal{P}_3 (a path of length 3).*

Next, let us define the following polynomials:

$$F_0(x) = 1, F_1(x) = x, F_2(x) = x^2 - k;$$

$$G_0(x) = 1, G_1(x) = x + 1;$$

$$H_{-2}(x) = -\frac{1}{k-1}, H_{-1}(x) = 0, H_0(x) = 1, H_1(x) = x;$$

$$P_{i+1}(x) = xP_i(x) - (k-1)P_{i-1}(x) \text{ for } \begin{cases} i \geq 2, & \text{if } P = F, \\ i \geq 1, & \text{if } P = G, \\ i \geq 1, & \text{if } P = H. \end{cases} \quad (2)$$

In [15], Singleton gives many relationships between these polynomials. We will use two of them. Given any $i \geq 0$,

$$G_i(x) = \sum_{j=0}^i F_j(x) \quad (3)$$

$$G_{i+1}(x) + (k-1)G_i(x) = (x+k)H_i(x). \quad (4)$$

The above defined polynomials have a close connection to the properties of a graph G . Namely, for $t < g$ the element $(F_t(A))_{x,y}$ counts the number of paths of length t joining vertices x and y of G . It follows from (3) that $G_t(A)$ counts the number of paths of length at most t joining pairs of vertices in G . All of the preceding claims can be found in [5]. The next lemma is based on the structure of G described in Lemma 2.1:

Lemma 2.2 *Let $k \geq 6$, $g = 2d \geq 6$ and let G be a (k, g) -graph of excess 4. If A is the adjacency matrix of G and E is the excess matrix of G , then*

$$F_d(A) = kA_d - AE.$$

Proof. Let $f = \{u, v\}$ be a base edge of the Moore tree and let $f_1 = \{w_1, w_2\}$, $f_2 = \{w_3, w_4\}$ be the edges of the subgraph induced by X_f . Also, let us assume that $d(u, w_1) = d(u, w_3) = d$ and $d(u, w_2) = d(u, w_4) = d+1$. We consider the case when $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$ in which case the excess vertices do not share common neighbour. The other cases when $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$ and the excess vertices share common neighbour or the subgraph induced by the excess vertices contains \mathcal{P}_3 are analogous. Since there are $k-1$ paths of length d from u to w_1 and w_3 , by the definition of $F_i(x)$ we have $(F_d(A))_{u,w_1} = (F_d(A))_{u,w_3} = k-1$. Considering the vertices of distance d from u , there are also the $(k-1)^{d-1}$ leaves of the subtree rooted at v . For $2(k-1)$ of these vertices there exists $k-1$ paths of length d from u to them. Namely, they are the vertices adjacent to w_2 or w_4 . For all the other leaves, there are k paths between. Thus, $(F_d(A))_{u,s} = 0$ if $d(u, s) \neq d$, $(F_d(A))_{u,s} = k$ if s is a leaf of a branch rooted at v and not adjacent to w_2 and w_4 , and $(F_d(A))_{u,s} = k-1$ if s is w_1, w_3 or a leaf of a branch rooted at v and adjacent to w_3 or w_4 . This yields for the matrix kA_d that $(kA_d)_{u,s} = k$ if $d(u, s) = d$ and $(kA_d)_{u,s} = 0$ if $d(u, s) \neq d$. Now, let s be a vertex of G such that $d(u, s) = d$ and s is adjacent to w_2 or w_4 . If $s = w_1$ or $s = w_3$ then it is easy to see that $(AE)_{u,s} = 1$. On the other hand, since s is adjacent to the subtree rooted at u through $k-2$ different horizontal edges, it follows that between the $k-1$ branches of the subtree rooted at u there exists one sub-branch that is not adjacent to s through a horizontal edge. Let s_1 be the root of that sub-branch. Then, $d(s, s_1) = d+1$, $d(u, s_1) = 1$, which implies $(A)_{u,s_1} = 1$ and $(E)_{s_1,s} = 1$. Let s_2 be the other vertex of distance $d+1$ from s . Because all neighbours of u , except s_1 , are of distance smaller than $d+1$ of s , we have $(A)_{u,s_2} = 0$ and $(E)_{s_2,s} = 1$. Thus $(AE)_{u,s} = 1$. If s is a vertex of G such that $d(u, s) = d$ and s is not adjacent to w_2 or w_4 then the distance between s and the neighbours of u is $d-1$. In this case, $(AE)_{u,s} = 0$. If $d(u, s) \neq d$ then the distance between s and the neighbours of u is different from $d+1$, and therefore $(AE)_{u,s} = 0$. The required identity follows from summing up the above conclusions. *q.e.d.*

Lemma 2.3 *Let $k \geq 6$, $g = 2d \geq 6$ and let G be a (k, g) -graph of excess 4. If A is the adjacency matrix of G , E is the excess matrix of G and J is the all-ones matrix,*

then

$$kJ = (A + kI)(H_{d-1}(A) + E).$$

Proof. By the definition of the polynomials $G_i(x)$ and using the fact that G has diameter $d + 1$ we conclude $J = G_{d-1}(A) + A_d + E$. The relation (3), setting $i = d$, asserts $G_d(A) = G_{d-1}(A) + F_d(A)$. Substituting this identity in (4), where we fix $i = d - 1$, we get $kG_{d-1}(A) + F_d(A) = (A + kI)H_{d-1}(A)$. Due to Lemma 2.2 the last identity is equivalent to $kG_{d-1}(A) + kA_d + kE = (A + kI)(H_{d-1}(A) + E)$. From $kJ = kG_{d-1}(A) + kA_d + kE$ follows $kJ = (A + kI)(H_{d-1}(A) + E)$. *q.e.d.*

The next theorem gives a relationship between the eigenvalues of the matrices A and E (this result is an analogue of Theorem 3.1 in [5]):

Theorem 2.4 *If $\mu (\neq \pm k)$ is an eigenvalue of A , then*

$$H_{d-1}(\mu) = -\lambda,$$

where λ is an eigenvalue of E .

Proof. Let us suppose that μ is an eigenvalue of A . Since G is a k -regular graph, the all-ones matrix J is a polynomial in A . This implies that any eigenvector of A is also an eigenvector of J . From $kJ = (A + kI)(H_{d-1}(A) + E)$ and since $H_{d-1}(A)$ is also a polynomial in A , we have that E is a polynomial in A , and consequently, every eigenvector of A is an eigenvector of E . Therefore, the eigenvalues of kJ are of the form $(\mu + k)(H_{d-1}(\mu) + \lambda)$. As is well-known, the eigenvalues of kJ are kn (with multiplicity 1) and 0 (with multiplicity $n - 1$). The eigenvalue kn corresponds to $\mu = k$, and so all the remaining eigenvalues, except for $-k$, satisfy the above equation. *q.e.d.*

Since the eigenvalues of a disjoint union of cycles are known, we are now in a position to determine the spectrum of A :

Lemma 2.5 *Let $k \geq 6$, $g = 2d \geq 6$ and let G be a (k, g) -graph of excess 4. If A is the adjacency matrix of G and E is the excess matrix of G , then:*

- 1) *The matrix E is the adjacency matrix of a graph $G(E)$, consisting of a disjoint union of c cycles C_i of length l_i with $1 \leq i \leq c$. Moreover, if d is odd and V_1 and V_2 are the two partition sets of the bipartite graph G , then every cycle in $G(E)$ is completely contained either in V_1 or V_2 .*
- 2) *The spectrum of A consists of:*
 - 2.1) $\pm k$, $c - 2$ many solutions of $H_{d-1}(x) = -2$, and one solution of each equation $H_{d-1}(x) = -2 \cos(\frac{2\pi j}{l_i})$, $j = 1, \dots, l_i - 1$; $1 \leq i \leq c$, for d odd;
 - 2.2) $\pm k$, $c - 1$ many solutions of $H_{d-1}(x) = -2$, and one solution of each equation (except one) $H_{d-1}(x) = -2 \cos(\frac{2\pi j}{l_i})$, $j = 1, \dots, l_i - 1$; $1 \leq i \leq c$, for d even;

Proof. 1) Our proof is analogous to that of Kovács for girth 5, [12], and Garbe's proof for odd girth $g = 2k + 1 > 5$, [10]. Let $f = \{u, v\}$ be a base edge of a bipartite Moore tree of G . Lemma 2.1 asserts that there exist exactly two vertices of G on distance $d + 1$ from u . Namely, they are the excess vertices adjacent to the leaves of the subtree rooted at v . The excess matrix E is the adjacency matrix for the graph $G(E)$ with same vertex set V as G such that two vertices of $G(E)$ are adjacent if and only if they are of distance $d + 1$. Because for each vertex $u \in V(G)$ there are exactly two vertices on distance $d + 1$ from u , every component of $G(E)$ is a cycle. Let c be the number of these cycles and let $l_i, i = 1, \dots, c$, be the lengths of these cycles ordered in an arbitrary manner. Moreover, if d is an odd number, any two vertices of G with distance $d + 1$ lie in the same partite set. Therefore any connected component of $G(E)$ is entirely contained either in V_1 or V_2 .

2) The eigenvalues of an n -cycle are known and are equal to $2 \cos(\frac{2\pi j}{n})$, ($j = 0, \dots, n - 1$). Therefore the eigenvalues of $G(E)$ are $2 \cos(\frac{2\pi j}{l_i})$, $j = 0, 1, \dots, l_i - 1$; $1 \leq i \leq c$, [10]. Since G is a k -regular bipartite graph, it has (among others) the eigenvalues k and $-k$. Let V_1 and V_2 be the partition sets of G . Hence the eigenvector of A corresponding to k consist of the all-ones vector j , and the eigenvector corresponding to $-k$ is the vector j' with values 1 on V_1 and values -1 on V_2 . If d is an odd number then two vertices of $G(E)$ are adjacent if and only if they are in the same partite set. Therefore $E \cdot j' = 2j'$, which implies that from the set of c solutions on $H_{d-1}(x) = -2$ we need to subtract two multiplicities for the eigenvalues k and $-k$ of G . If d is an even number then two vertices of $G(E)$ are adjacent if and only if they are in different partite sets. Thus $E \cdot j' = -2j'$. In this case, from the set of c solutions on $H_{d-1}(x) = -2$ we need to subtract one multiplicity for the eigenvalue k and from the set of all solutions on $H_{d-1}(x) = 2$ we need to subtract one multiplicity for the eigenvalue $-k$. *q.e.d.*

Lemma 2.6 *Let $k \geq 6$, $g = 2d \geq 6$ and let G be a (k, g) -graph of excess 4. Furthermore, let c be the number of cycles of $G(E)$ and c_2 be the number of cycles of even length. Then:*

- 1) *If $H_{d-1}(x) - 2$ is irreducible over $\mathbb{Q}[x]$ then $d - 1$ divides $c - 1$ or $c - 2$;*
- 2) *If $H_{d-1}(x) + 2$ is irreducible over $\mathbb{Q}[x]$, then $d - 1$ divides $c_2 - 1$ or c_2 .*

Proof. 1) Combining Theorem 2.4 and part 2) from Lemma 2.5 we obtain that $H_{d-1}(x) - 2$ is an irreducible factor of the characteristic polynomial of A . Realizing that the roots of an irreducible factor of a characteristic polynomial of given rational symmetric matrix have the same multiplicities, [12], from 2) of Lemma 2.5 we have: If d is an even number then the $d - 1$ roots of $H_{d-1}(x) - 2$ have multiplicity $\frac{c-1}{d-1}$, which has to be a positive integer. If d is odd then the $d - 1$ roots have multiplicity $\frac{c-2}{d-1}$.

2) Part 2) follows along the same lines as part 1). *q.e.d.*

We can base the testing of irreducibility of $H_{d-1}(x) \pm 2$ on the well-known Eisenstein's criterion that asserts for a polynomial $f(x) = \sum_{i=0}^n a_i x^i \in \mathbb{Z}[x]$ and a prime p that divides a_i for all $0 \leq i < n$, does not divide a_n and p^2 does not divide a_0 . Now we are ready for the main result in this section:

Theorem 2.7 *Let $k \geq 7$ be an odd number and let $g = 2d \geq 8$. Let c be the number of cycles of $G(E)$ and c_2 be the number of cycles with even length. If there exists a (k, g) -graph of excess 4 then*

- 1) *if d is an odd number then $d - 1$ divides $c - 2$ and c_2 ;*
- 2) *if d is an even number then $d - 1$ divides $c - 1$ and $c_2 - 1$.*

Proof. According to Lemma 2.6, it is enough to prove that the polynomials $H_{d-1}(x) - 2$ and $H_{d-1}(x) + 2$ are irreducible. We will prove using induction on $d \geq 4$ that $H_{d-1}(x) = x^{d-1} + (k-1) \cdot P_{d-3}(x)$, where $P_{d-3}(x)$ is an integer polynomial of degree $d-3$. For $d = 4$ we calculate $H_3(x) = x^3 - 2(k-1)x$. Let us suppose that the above formula holds for $H_{d-2}(x)$ and $H_{d-3}(x)$. That yields

$$\begin{aligned} H_{d-1}(x) &= x(x^{d-2} + (k-1) \cdot P_{d-4}(x)) - (k-1)(x^{d-3} + (k-1) \cdot P_{d-5}(x)) = \\ &= x^{d-1} + (k-1) \cdot P_{d-3}(x). \end{aligned}$$

Therefore $H_{d-1}(x) \pm 2 = x^{d-1} + (k-1) \cdot P_{d-3}(x) \pm 2$. By the inductive hypothesis, follows that for an odd d occurs $H_{d-1}(0) = (-1)^{\frac{d-1}{2}} \cdot (k-1)^{\frac{d-1}{2}}$ and $H_{d-1}(0) = 0$ for an even d . Hence for an odd $d \geq 5$ the absolute value $(-1)^{\frac{d-1}{2}} \cdot (k-1)^{\frac{d-1}{2}} \pm 2$ is not divisible by 2^2 , and clearly for an even $d \geq 4$, ± 2 is not divisible by 2^2 . Since $k-1$ is even, it follows that every coefficient on $H_{d-1}(x) \pm 2$ except for the coefficient 1 of x^{d-1} is divisible by 2. Thus, the conditions of the Eisenstein's criterion are satisfied, and $H_{d-1}(x) \pm 2$ is irreducible. *q.e.d.*

3 The non-existence of bipartite graphs of cyclic or bicyclic excess

In this section we still deal with the family of graphs considered as in Section 2. Again, let $k \geq 6, g = 2d \geq 6$ and let G be a (k, g) -graph of excess 4 and order n . Clearly n is even number. We have already proved that the excess graph $G(E)$ consists of a disjoint union of c cycles $C_i, 1 \leq i \leq c$. If $c = 1$ and $G(E)$ consists of an n -cycle, G is of cyclic excess 4, and if $c = 2$ and $G(E)$ consists of a disjoint union of two cycles, G is of bicyclic excess 4. These are the graphs we study in this section. Note that there are no graphs G with cyclic excess 4 if d is an odd number; in this case we showed that each cycle of $G(E)$ is completely contained either in V_1 or V_2 .

Let d be an even number and let L_n be an n -cycle formed by the vertices of $G(E)$. If A' is the adjacency matrix of L_n , its characteristic polynomial $\chi(L_n, x)$ satisfies $\chi(L_n, x) = (x-2)(x+2)(R_n(x))^2$, where R_n is a monic polynomial of degree

$\frac{n}{2} - 1$. Consider the factorization $x^n - 1 = \prod_{l|n} \Phi_l(x)$, where $\Phi_l(x)$ denotes the l -th cyclotomic polynomial. In the following paragraph, we summarize the properties of cyclotomic polynomials as listed in [6].

The cyclotomic polynomial $\Phi_l(x)$ has integral coefficients, it is irreducible over $\mathbb{Q}[x]$, and it is *self-reciprocal* ($x^{\phi(l)}\Phi_l(1/x) = \Phi_l(x)$). From the irreducibility and the self-reciprocity of $\Phi_l(x)$ follows that the degree of $\Phi_l(x)$ is even for $l \geq 2$.

Thus we obtain the following factorization of $R_n(x)$: $R_n(x) = \prod_{3 \leq l|n} f_l(x)$, where f_l is an integer polynomial of degree $\frac{\phi(l)}{2}$ satisfying $x^{\phi(l)/2}f_l(x + 1/x) = \Phi_l(x)$. Also, f_l is irreducible over $\mathbb{Q}[x]$ and $f_3(x) = x + 1$, $f_4(x) = x$, $f_5(x) = x^2 + x - 1$, $f_6(x) = x - 1$. Substituting $y = -H_{d-1}(x)$ into $\frac{\chi(L_n, y)}{(y-2)}$, we obtain a polynomial $F(x)$ of degree $(n-1)(d-1)$ which satisfies $F(A)u = 0$ for each eigenvector u of A orthogonal to the all -1 vector. Setting $F_{l,k,d-1}(x) = f_l(-H_{d-1}(x))$ yields

$$F(x) = (-H_{d-1}(x) + 2) \prod_{3 \leq l|n} (F_{l,k,d-1}(x))^2.$$

Lemma 3.1 *Let $g = 2d > 6$ and $l \geq 3$ be a divisor of n . If there is a (k, g) -graph with cyclic excess 4 and order n , then $F_{l,k,d-1}(x)$ must be reducible over $\mathbb{Q}[x]$.*

Proof. The degree of $F_{l,k,d-1}(x)$ is equal to $(d-1) \cdot \frac{\phi(l)}{2}$. If $F_{l,k,d-1}(x)$ is irreducible over $\mathbb{Q}[x]$, then all its roots must be eigenvalues of A . Employing Observation 3.1. from [6], we conclude that there are at most $\phi(l)$ roots of $F_{l,k,d-1}(x)$ that are eigenvalues of A . Thus $(d-1) \cdot \frac{\phi(l)}{2} = \phi(l)$ i.e., $d = 3$. This contradicts the assumption that $2d > 6$. *q.e.d.*

Note that $\deg(F_{l,k,d-1}(x)) = d-1$ if and only if $\phi(l) = 2$, i.e., if and only if $l \in \{3, 4, 6\}$.

Lemma 3.2 *Let $k \geq 6$, $g = 2d > 6$, and let n be the order of a (k, g) -graph with cyclic excess 4. Then*

- 1) *if $n \equiv 0 \pmod{3}$, then $H_{d-1}(x) - 1$ must be reducible over $\mathbb{Q}[x]$;*
- 2) *if $n \equiv 0 \pmod{4}$, then $H_{d-1}(x)$ must be reducible over $\mathbb{Q}[x]$;*
- 3) *if $n \equiv 0 \pmod{6}$, then $H_{d-1}(x) + 1$ must be reducible over $\mathbb{Q}[x]$.*

Proof. Follows directly from Lemma 3.1, with the additional assumption $f_3(x) = x + 1$, $f_4(x) = x$ and $f_6(x) = x - 1$. *q.e.d.*

If $n \equiv 0 \pmod{4}$, then using the formula for the order of G , $d-1$ must be odd. On the other hand, since $H_1(x) = x$, $H_3(x) = x^3 - 2(k-1)x$ and $H_{d-1}(x) = xH_{d-2}(x) - (k-1)H_{d-3}(x)$ we see that if $d-1$ is an odd number then x divides $H_{d-1}(x)$, which implies that $H_{d-1}(x)$ is reducible. Therefore the second condition from Lemma 3.2 is satisfied.

The irreducibility of the polynomials $H_{d-1}(x) - 1$ over $\mathbb{Q}[x]$ is examined in [5], where it is analytically proven that these polynomials are irreducible for $d \in \{4, 6, 8\}$ and the paper contains a conjecture that $d \geq 10$, $H_{d-1}(x) - 1$ is irreducible. From the irreducibility of $H_{d-1}(x) - 1$ we obtain the main non-existence result of our paper.

Theorem 3.3 *If k and g satisfy one of the following conditions, there exist no (k, g) -graphs of cyclic excess 4:*

- 1) $k \equiv 1, 2 \pmod{3}$ and $g = 8$;
- 2) $k \equiv 1 \pmod{3}$ and $g = 12$;
- 3) $k \equiv 1 \pmod{3}$ and $g = 16$.

Proof. Because the order of the graphs is equal to $4+2(1+(k-1)+\dots+(k-1)^{(g-2)/2})$ we conclude $n \equiv 0 \pmod{3}$. Since the polynomial $H_{d-1}(x) - 1$ is known to be irreducible for $d \in \{4, 6, 8\}$, we get contradiction to 1) from Lemma 3.2. *q.e.d.*

Remark: Since d is an even number, Theorem 2.7 asserts that $d - 1$ divides $c - 1$ and $c_2 - 1$. This claim is satisfied because $c = c_2 = 1$.

Next, let us consider graphs of bicyclic excess 4. In this case, we can assume an arbitrary (even or odd) d , as this case does not depend of the parity of d . So, let $G(E)$ be a graph consisting of a disjoint union of two cycles C_1 and C_2 . If d is an odd number, then the vertex sets of the cycles C_1 and C_2 correspond to the partite sets V_1 and V_2 , respectively.

If $n \equiv 0 \pmod{4}$, d is an even, each edge of $C(E)$ has endpoints in V_1 and V_2 , and therefore each of the cycles has even length, $c_2 = 2$. Furthermore, $k - 1$ must be odd. Unfortunately, this will not help us in excluding any family of pairs (k, g) for which G does not exist. In fact, for an odd $d - 1$ and an odd $k - 1$ we cannot conclude irreducibility of $H_{d-1}(x) + 2$, thus, we cannot employ Lemma 2.6.

If $n \equiv 2 \pmod{4}$ and d is odd, then the lengths of C_1 and C_2 are equal to $\frac{n}{2}$ (clearly $n = 2s + 1$ is odd). Therefore $c_2 = 0$ and clearly $d - 1$ divides $c - 2$ and c_2 .

The main result about the non-existence of graphs G with bicyclic excess 4 is given in the following theorem:

Theorem 3.4 *If $k \geq 7$ is an odd and $g = 2d \geq 8$, where d is an even integer, then there exist no (k, g) -graphs with bicyclic excess 4.*

Proof. We have $c = 2$. Theorem 2.7 implies that $d - 1$ divides $c - 1$; a contradiction. *q.e.d.*

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